LA-UR-19-29722

Title: FULLY LOCAL QUASIDIFFUSION

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Intended for: Report

Issued: 2019-09-26

Fully Local Quasidiffusion

End of Summer Presentation

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August 13, 2019

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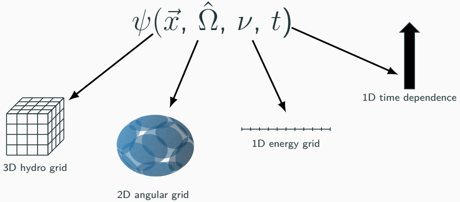
Background

Thermal Radiative Transfer

- Describes conservation and transfer of energy between photons and matter
- 6+1 dimensional phase space $\Rightarrow$ dominates memory and runtime
- Capsaicin: *algorithmic* improvements are all that's left
- Goal: first steps toward a QD TRT algorithm with cell-local coupling to the material energy balance equation



Quasidiffusion/Variable Eddington Factor Method

- An old method with two names and three disambiguations
  - Reactor transport: Quasi-*diffusion*
  - Astrophysicists: VEF has been in our code for decades...
  - TRT transport:
    - Robust, non-linear acceleration scheme
    - Two-level in angle
    - Nonlinear projective iteration *not* additive correction
- Consistent Discretization
  - Discretized QD matches discretized transport exactly
- Inconsistent Discretization
  - Differ by discretization error
  - Acceleration properties the same given QD terms are properly represented
- QD algorithms can take advantage of this flexibility

The Bleeding Edge of QD

- Anistratov and Warsa (NSE, 2018)
  - Consistent linear-linear DG discretization
  - Compared many types of cell interface conditions
- Warsa and Anistratov (JCTT, 2018)
  - Inconsistent discretization can affect acceleration properties
  - NKA recovers iterative efficiency
- Anistratov, Warsa, and Lowrie (M&C 2019)
  - Investigated processor-local QD with processor boundary conditions to accelerate PBJ

All were 1D only

Proposed Algorithm

- Combine them all and implement in Capsaicin on 2D triangles
- Extend processor-local QD to Fully Local
  - Solve each cell independently
  - Compute interior boundary conditions from angular flux that decouple the cells
- Use NKA to retain iterative efficiency
- TRT algorithm: nonlinearly iterate FLQD with material energy balance equation "at the bottom"
  - Use FLQD as an inexpensive proxy for transport
  - Fully Local $\Rightarrow$ Newton iterations don't require a global solve
  - Cell-wise coupling better than point-wise

Show scheme is reasonably effective for Linear Transport before TRT

Moment Equations

- Steady-state, mono-energetic, isotropic scattering and source

$$\hat{\Omega} \cdot \nabla \psi + \sigma_t \psi = \frac{\sigma_s}{4\pi} \int \psi \, \mathrm{d}\Omega' + \frac{Q}{4\pi}$$

- Angular moments always have more unknowns than equations

$$\nabla \cdot \vec{J} + \sigma_a \phi = Q \,,$$

$$\nabla \cdot \mathbf{P} + \sigma_t \vec{J} = 0$$

with

$$\phi = \int \psi \, \mathrm{d}\Omega \,, \quad \vec{J} = \int \hat{\Omega} \, \psi \, \mathrm{d}\Omega \,, \quad \mathbf{P} = \int \hat{\Omega} \otimes \hat{\Omega} \, \psi \, \mathrm{d}\Omega$$

- 3D $\Rightarrow$ 6 + 3 + 1 = 10 unknowns with only 4 equations

The Philosophy of QD

- When in doubt, multiply and divide by the scalar flux

$$\mathbf{P} = \int \hat{\Omega} \otimes \hat{\Omega}\, \psi \,\mathrm{d}\Omega \rightarrow \underbrace{\frac{\int \hat{\Omega} \otimes \hat{\Omega}\, \psi \,\mathrm{d}\Omega}{\int \psi \,\mathrm{d}\Omega}}_{\mathbf{E}} \phi = \mathbf{E}\phi$$

- QD equations:

$$\nabla \cdot \vec{J} + \sigma_a \phi = Q$$
$$\nabla \cdot (\mathbf{E}\phi) + \sigma_t \vec{J} = 0$$

- ψ linearly anisotropic $\Rightarrow \mathbf{E} = \frac{1}{3}\mathbf{I}$, Fick's Law
- Tensor diffusion in first-order form

$$\nabla \cdot \vec{J} + \sigma_a \phi = Q$$
$$\mathbf{D} \cdot \nabla \phi + \vec{J} = 0$$

- QD second-order form has all mixed derivatives in addition to Laplacian terms $\Rightarrow$ difficult to discretize

Linear Transport QD Algorithm

- Solve

$$\hat{\Omega} \cdot \nabla \psi^{\ell+1/2} + \sigma_t \psi^{\ell+1/2} = \frac{\sigma_s}{4\pi} \phi^\ell + \frac{Q}{4\pi}$$

  for $\psi^{\ell+1/2}$

- Compute Eddington tensor:

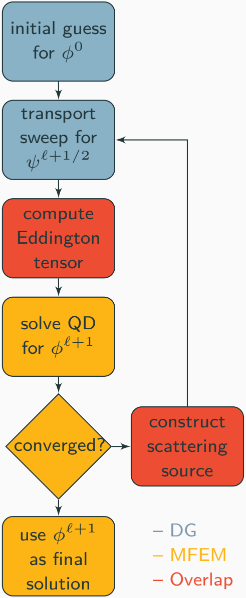
$$\mathbf{E}^{\ell+1/2} = \frac{\sum w_d \, \hat{\Omega}_d \otimes \hat{\Omega}_d \, \psi_d^{\ell+1/2}}{\sum w_d \psi_d^{\ell+1/2}}$$

- Solve QD equations for updated scalar flux $\phi^{\ell+1}$

$$\nabla \cdot \vec{J}^{\ell+1} + \sigma_a \phi^{\ell+1} = Q \,,$$

$$\nabla \cdot \left(\mathbf{E}^{\ell+1/2} \phi^{\ell+1} \right) + \sigma_t \vec{J}^{\ell+1} = 0 \,.$$

- Update scattering term with QD solution
- Stop when $\|\phi^{\ell+1} - \phi^\ell\| < tol$



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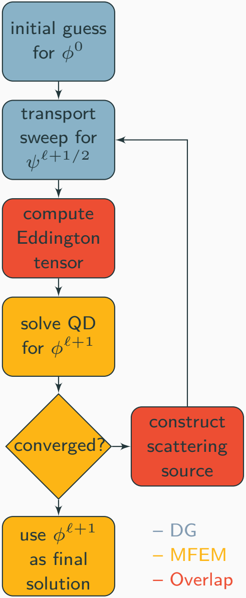


Properties

- Acceleration occurs because Eddington factors converge quickly
  - Depends on angular shape not magnitude
  - ψ converges quickly in angular shape
  - Compensates lagging of scattering term in Source Iteration

Discretization

Zeroth Moment

$$\boxed{\nabla \cdot \vec{J} + \sigma_a \phi = Q}$$

- Anistratov and Warsa consistent discretization has both ϕ and $\vec{J}$ approximated with linear DG
- Multiply by test function u and integrate over single element

$$\int u \nabla \cdot \vec{J} \mathrm{d}V + \int \sigma_a \, u\phi \, \mathrm{d}V = \int u \, Q \, \mathrm{d}V$$

- Integrate by parts since $\vec{J}$ is discontinuously approximated

$$\oint u \, \widehat{J}_n \, \mathrm{d}A - \int \nabla u \cdot \vec{J} \mathrm{d}V + \int \sigma_a \, u\phi \, \mathrm{d}V = \int u \, Q \, \mathrm{d}V$$

where $\widehat{J}_n$ is an upwind-consistent net current

FEM Interpolation

- Shape functions on reference triangle

$$B_1(\xi, \eta) = 1 - \xi - \eta\,, \quad B_2(\xi, \eta) = \xi\,, \quad B_3(\xi, \eta) = \eta$$

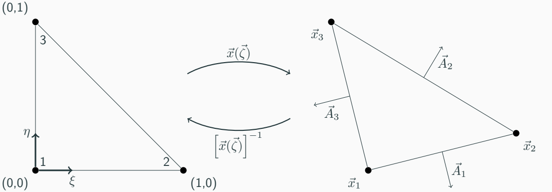
- Interpolate scalar flux with linear combination of shape functions

$$\phi(\xi, \eta) = \sum_j B_j(\xi, \eta)\phi_j$$

- Re-write as dot product of vectors of shape functions and coefficients

$$\phi(\xi, \eta) = \begin{bmatrix} B_1(\xi, \eta) & B_2(\xi, \eta) & B_3(\xi, \eta) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$$= \mathbf{B}\underline{\phi}$$



FEM Interpolation (cont.)

- Interpolate each component of the current with linear shape functions

$$J_d(\xi,\eta) = \sum_j B_j(\xi,\eta) J_{d,j}\,, \quad d = x, y$$

- Re-write as matrix-vector product

$$\vec{J}(\xi,\eta) = \begin{bmatrix} B_1 & B_2 & B_3 & & & \\ & & & B_1 & B_2 & B_3 \end{bmatrix} \begin{bmatrix} J_{x1} \\ J_{x2} \\ J_{x3} \\ J_{y1} \\ J_{y2} \\ J_{y3} \end{bmatrix}$$

$$= \mathbf{N}\underline{J}$$

Discrete Zeroth Moment

$$\boxed{-\int \nabla u \cdot \vec{J}\,\mathrm{d}V + \int \sigma_a\, u\phi\,\mathrm{d}V = \int u\,Q\,\mathrm{d}V - \oint u\,\widehat{J}_n\,\mathrm{d}A}$$

$$-\mathbf{D}\underline{J} + \mathbf{M}_a\underline{\phi} = \mathbf{M}\underline{Q} - \underline{J}_b$$

with

$$\mathbf{D} = \int (\nabla\mathbf{B})^T\,\mathbf{N}\,\mathrm{d}V\,, \quad \mathbf{M}_a = \int \sigma_a\mathbf{B}^T\mathbf{B}\,\mathrm{d}V\,,$$

$$\mathbf{M} = \int \mathbf{B}^T\mathbf{B}\,\mathrm{d}V$$

Use isoparametric transformation to transform derivatives and convert from reference to physical space

First Moment

$$\boxed{\nabla \cdot (\mathbf{E}\phi) + \sigma_t \vec{J} = 0}$$

- Multiply by vector-valued test function $\vec{v}$ and integrate over element

$$\int \vec{v} \cdot \nabla \cdot (\mathbf{E}\phi)\,\mathrm{d}V + \int \sigma_t\, \vec{v} \cdot \vec{J}\,\mathrm{d}V = 0$$

- Integrate by parts since both ϕ and $\mathbf{E}$ are discontinuous

$$\oint \vec{v} \cdot \mathbf{E} \cdot \hat{n}\, \widehat{\phi}\,\mathrm{d}V - \int \nabla\vec{v} : \mathbf{E}\,\phi\,\mathrm{d}V + \int \sigma_t\, \vec{v} \cdot \vec{J}\,\mathrm{d}V = 0$$

where

$$\nabla\vec{v} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{bmatrix} ,$$

$$\mathbf{A} : \mathbf{B} = \sum_i \sum_j A_{ij} B_{ij} = A_{11}B_{11} + A_{12}B_{12} + A_{21}B_{21} + A_{22}B_{22}$$

and $\widehat{\phi}$ is an upwind-consistent scalar flux

Discrete First Moment

$$\boxed{-\int \nabla\vec{v} : \mathbf{E}\,\phi\,\mathrm{d}V + \int \sigma_t \vec{v}\cdot\vec{J}\,\mathrm{d}V = -\oint \vec{v}\cdot\mathbf{E}\cdot\hat{n}\,\widehat{\phi}\,\mathrm{d}A}$$

$$-\mathbf{G}\underline{\phi} + \mathbf{M}_t\underline{J} = -\underline{\phi}_b$$

with

$$\mathbf{G} = \int (\nabla\mathbf{N})^T\,\mathbf{E}\mathbf{B}\,\mathrm{d}V\,, \quad \mathbf{M}_t = \int \sigma_t \mathbf{N}^T\mathbf{N}\,\mathrm{d}V$$

Boundary Terms

Upwind Consistency

- Want consistency with DG transport
- Face between elements e and e' with normal from $e \to e'$, the upwind angular flux is

$$\hat{\Omega} \cdot \hat{n}\, \widehat{\psi} = \frac{1}{2}\Big(|\hat{\Omega} \cdot \hat{n}| + \hat{\Omega} \cdot \hat{n}\Big)\, \psi_e + \frac{1}{2}\Big(|\hat{\Omega} \cdot \hat{n}| - \hat{\Omega} \cdot \hat{n}\Big)\, \psi_{e'}$$

- Discrete current

$$\int \hat{\Omega} \cdot \hat{n}\, \widehat{\psi}\, \mathrm{d}\Omega = \underbrace{\int_{\hat{\Omega}\cdot\hat{n}>0} \hat{\Omega} \cdot \hat{n}\, \psi_e \, \mathrm{d}\Omega}_{\text{local outflow}} + \underbrace{\int_{\hat{\Omega}\cdot\hat{n}<0} \hat{\Omega} \cdot \hat{n}\, \psi_{e'} \, \mathrm{d}\Omega}_{\text{neighbor's inflow}}$$

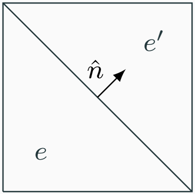
$$= J^+_{n,e} + J^-_{n,e'}$$

- Consistent boundary current:

$$\widehat{J}_n = J^+_{n,e} + J^-_{n,e'}$$

- Boundary scalar flux from half-range scalar fluxes

$$\widehat{\phi} = \phi^+_{n,e} + \phi^-_{n,e'}$$



Fully Local Discretization

- All terms are cell-local except boundary terms
- Boundary terms have a
  - Local outflow contribution ($J_{n,e}^{+}$ and $\phi_{n,e}^{+}$)
  - Non-local inflow contribution ($J_{n,e'}^{-}$ and $\phi_{n,e'}^{-}$)
- Decouple cells
  - Outflow from local information + QD BCs
  - Inflow from neighbor's high-order solution from previous sweep
- Solve QD equations on each cell *independently*

Miften-Larsen QD Outflow Condition

- Miften-Larsen QD boundary conditions:

$$
\begin{aligned}
\vec{J}\cdot\hat{n} &= J_n^+ + J_n^- \\
&= 2J_n^+ - (J_n^+ - J_n^-) \\
&= 2J_n^+ - \int |\hat{\Omega}\cdot\hat{n}|\,\psi\,\mathrm{d}\Omega \\
&= 2J_n^+ - \frac{\int |\hat{\Omega}\cdot\hat{n}|\,\psi\,\mathrm{d}\Omega}{\int \psi\,\mathrm{d}\Omega}\phi \\
&= 2J_n^+ - G\phi \\
\therefore J_n^+ &= \frac{1}{2}\left[\vec{J}\cdot\hat{n} + G\phi\right]
\end{aligned}
$$

- Provides expression for transport-consistent outflow partial current
- ψ linearly anisotropic $\Rightarrow G = \frac{1}{2}$, recover Marshak boundary conditions

High-Order Inflow Condition

- Compute inflow from ψ at previous iteration

$$J_n^- = \int_{\hat{\Omega}\cdot\hat{n}<0} \hat{\Omega}\cdot\hat{n}\,\psi\,\mathrm{d}\Omega$$

- Combining QD outflow and high-order inflow

$$\widehat{J}_n = \frac{1}{2}\left(\vec{J}\cdot\hat{n} + G\phi\right) + J_n^-$$

- Discrete boundary terms

$$\oint u\,\widehat{J}_n\,\mathrm{d}A \to \frac{1}{2}\oint \mathbf{B}^T\hat{n}^T\mathbf{N}\underline{J}\,\mathrm{d}A + \frac{1}{2}\oint G\mathbf{B}^T\mathbf{B}\underline{\phi}\,\mathrm{d}A + \oint \mathbf{B}^T\,J_n^-\,\mathrm{d}A$$

- Adds a bilinear form for the current and scalar flux and a RHS source term computed from transport

Scalar Flux QD Boundary

- Use high-order information to get a boundary form for $\phi_n^{\pm}$

$$\phi_n^+ = \frac{1}{C_n^+} J_n^+ \,, \quad C_n^+ = \frac{\int_{\hat{\Omega}\cdot\hat{n}>0} \hat{\Omega}\cdot\hat{n}\,\psi\,\mathrm{d}\Omega}{\int_{\hat{\Omega}\cdot\hat{n}>0} \psi\,\mathrm{d}\Omega}$$

- Combine with Miften-Larsen

$$\begin{aligned} \widehat{\phi} &= \phi_{n,e}^+ + \phi_{n,e'}^- \\ &= \frac{1}{C_n^+} J_n^+ + \phi_{n,e'}^- \\ &= \frac{1}{2C_n^+}\left[\vec{J}\cdot\hat{n} + G\phi\right] + \phi_n^- \end{aligned}$$

- Discrete first moment boundary term:

$$\oint \vec{v}\cdot\mathbf{E}\cdot\hat{n}\,\widehat{\phi}\,\mathrm{d}A \to \oint \mathbf{N}^T\mathbf{E}\hat{n}\,\frac{1}{2C_n^+}\hat{n}^T\mathbf{N}\underline{J}\,\mathrm{d}A + \oint \mathbf{N}^T\mathbf{E}\hat{n}\,\frac{G}{2C_n^+}\mathbf{B}\underline{\phi}\,\mathrm{d}A + \int \mathbf{N}^T\mathbf{E}\hat{n}\,\phi_n^-\,\mathrm{d}A$$

Putting it All Together

- For each element solve:

$$\begin{bmatrix} \mathbf{M}_a & -\mathbf{D} \\ -\mathbf{G} & \mathbf{M}_t \end{bmatrix} \begin{bmatrix} \underline{\phi} \\ \underline{J} \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

  with boundary terms included in the definitions of the left and right hand sides
- 9×9 system $\Rightarrow$ can directly invert
- Implemented with Trilinos' `Epetra_SerialDenseSolver`

Boundary Integration

Analytic Integration

- In Capsaicin, most FEM integrals are pre-evaluated by hand
- Problem: all QD terms inherit the spatial variance of $\psi(\vec{x}) = \mathbf{B}\underline{\psi}$

$$\mathbf{E}(\vec{x}) = \frac{\int \hat{\Omega} \otimes \hat{\Omega}\, \psi(\vec{x})\, \mathrm{d}\Omega}{\int \psi(\vec{x})\, \mathrm{d}\Omega} = \frac{\int \hat{\Omega} \otimes \hat{\Omega}\, \mathbf{B}\underline{\psi}\, \mathrm{d}\Omega}{\int \mathbf{B}\underline{\psi}\, \mathrm{d}\Omega}$$

$\Rightarrow$ $\mathbf{E}$, G, and C_n^+ are all *improper rational polynomials* in space

- Integral of improper rational polynomial involves logarithms of the denominator
  - $\ln \phi$ not defined for $\phi < 0 \Rightarrow$ loss of robustness to negativity
- First moment boundary terms are *exceptionally* complicated

$$\oint \mathbf{N}^T \mathbf{E} \hat{n} \frac{G}{2C_n^+} \mathbf{B}\, \mathrm{d}A = \oint \frac{\text{quintic polynomial}}{\text{cubic polynomial}}\, \mathrm{d}A$$

characterized by 18 coefficients (numerator and denominator of the QD terms at the two nodes on the face)

Method One: Closing "After the Fact"

- Apply an angular and spatial closure after discretizing in space

$$\oint \vec{v} \cdot \mathbf{E} \cdot \hat{n} \, \widehat{\phi} \, \mathrm{d}A \rightarrow \frac{\oint \vec{v} \cdot \mathbf{P} \cdot \hat{n}}{\oint \widehat{\phi} \, \mathrm{d}A} \oint \widehat{\phi} \, \mathrm{d}A$$

- After the fact closure avoids integrating rational polynomials
- Derive Miften-Larsen BCs in spatially discrete context, can get all terms as ratio of integrals instead of integral of ratios
- Motivation: in 1D the only integral is the gradient term (no boundary integrals)

$$\int \frac{\mathrm{d}B_i}{\mathrm{d}x} E B_j \, \mathrm{d}x \rightarrow \frac{\int \frac{\mathrm{d}B_i}{\mathrm{d}x} \int \mu^2 \, \psi \, \mathrm{d}\mu \, \mathrm{d}x}{\int \int \psi \, \mathrm{d}\mu \, \mathrm{d}x} \int B_j \, \mathrm{d}x$$

linear elements $\Rightarrow \frac{\mathrm{d}B_i}{\mathrm{d}x} =$ constant, after the fact closure is equivalent to using spatially averaged Eddington factor

Method Two: Numerical Quadrature

- Suboptimal convergence for discrete closure
- Analytic integrals are complicated, error prone, and have restrictions
- Implemented Gauss quadrature for the face integral terms involving QD factors
- After the fact may have worked in 1D because spatial averaging equivalent to one-point quadrature
    - Not possible to recast 2D boundary terms with discrete closure as low-order quadrature
- 2 point GQ is accurate enough
    - QD factors slowly vary in space for simple isotropic solutions

Results

Convergence Test

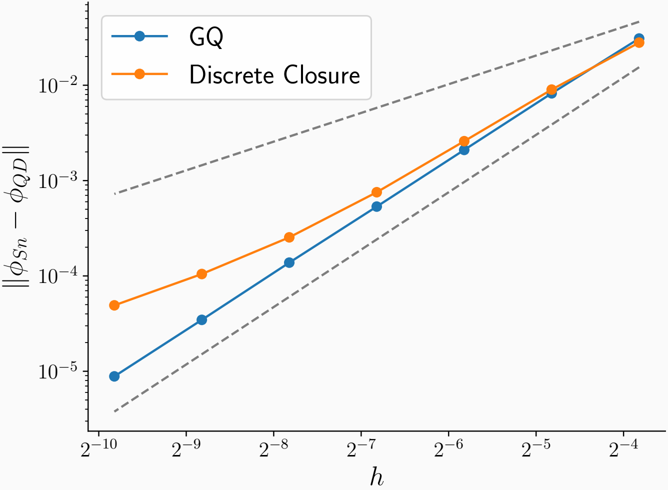
- MMS is somewhat difficult in Capsaicin
- Instead, use difference between transport solution and QD solution $+$ triangle inequality

$$
\begin{aligned}
\|\phi_{S_n} - \phi_{QD}\| &= \|(\phi_{S_n} - \phi) + (\phi - \phi_{QD})\| \\
&\leq \|\phi_{S_n} - \phi\| + \|\phi - \phi_{QD}\| \\
&= C_{S_n} h^2 + C_{QD} h^p \\
&\approx \mathcal{O}\left(h^{\min(2,p)}\right)
\end{aligned}
$$

- Use convergence of S$_n$ and QD as proxy for error
- Test can verify convergence up to second order
- All results used triangular Gauss-Chebyshev-Legendre S$_8$ and an iterative tolerance of 10^{-10}

Spatial Convergence Rates

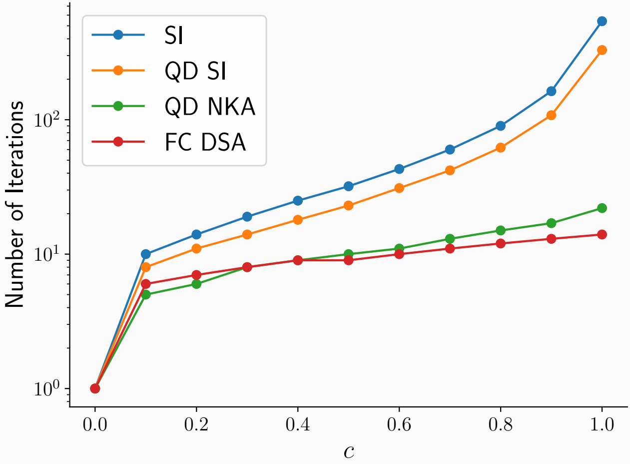
$$\vec{x} \in [0,1] \times [0,1]\,, \quad \sigma_t = \sigma_s = 4\,\mathrm{cm}^{-1}\,, \quad Q = 1\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$$



Fully Local QD second order for GQ only

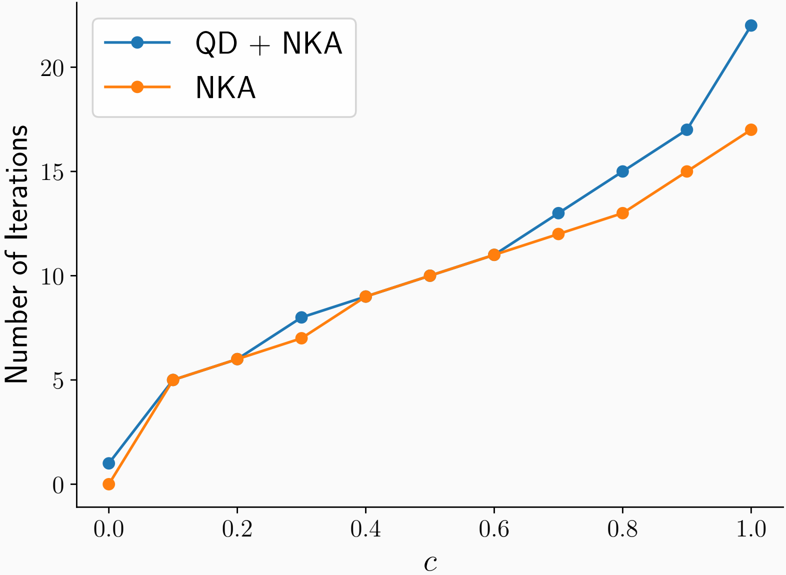
Scattering Ratio Test

$$\vec{x} \in [0,1] \times [0,1]\,, \quad \sigma_t = 10\,\mathrm{cm}^{-1}\,, \quad \sigma_s = c\sigma_t\,, \quad Q = 1\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$$



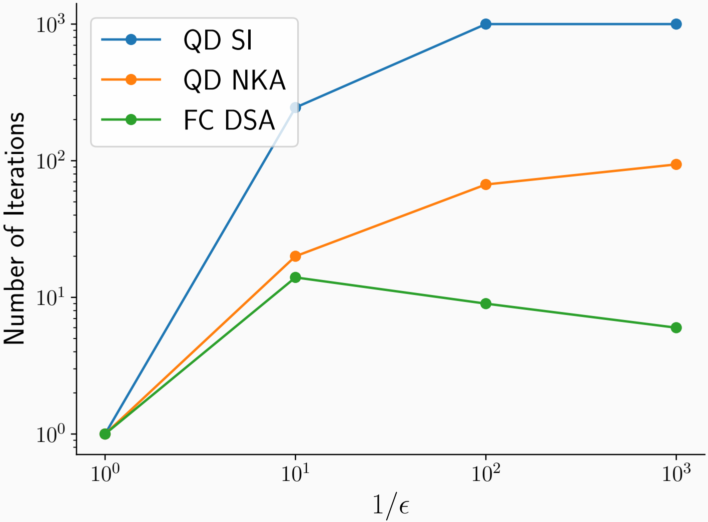
Fully Local + NKA similar to DSA

Is it just NKA?

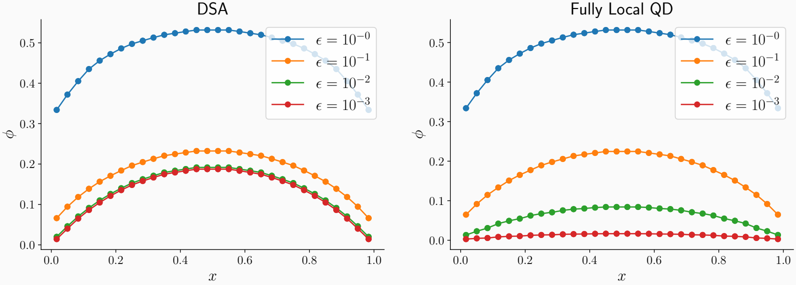


Thick Diffusion Limit

$$\vec{x} \in [0,1] \times [0,1]\,, \quad \sigma_t = \frac{1}{\epsilon}\,, \quad \sigma_s = \frac{1}{\epsilon} - \epsilon\,, \quad Q = \epsilon$$



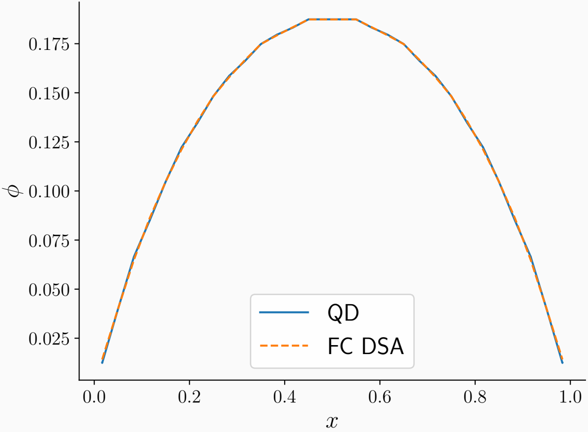
Thick Diffusion Limit (cont.)



Fully Local doesn't maintain thick diffusion limit

Thick Diffusion Limit (cont.)

- Converge DSA with $\epsilon = 10^{-3}$
- Use ψ to do one iteration of FLQD
- Compare scalar fluxes



FLQD still effective as a proxy in thick diffusion limit

Conclusions

- After the fact closure had suboptimal spatial accuracy not seen in 1D
- Numerical quadrature of rational polynomial terms led to second order algorithm
- Fully Local not effective for linear transport
  - Weak acceleration effects
  - Krylov just as effective
  - No thick diffusion limit
- Still viable for intended TRT use
  - At least second order accurate $\Rightarrow$ accurate proxy for cell-local coupling

Future Work

- MMS
  - FLQD could be third order
  - Is discrete closure inconsistent with transport or with QD equations?
- Implement global and processor local QD
  - Couple elements by computing J_n^- and ϕ_n^- from Miften-Larsen of neighbor cell
- Extend to parallel
  - Communicate QD factors across parallel boundaries
- Compare Fully Local, processor-local, and global QD for accelerating PBJ
- Accuracy and expense of numerical quadrature for anisotropic problems
- TRT

References

[1] D. Anistratov and J. Warsa, *Discontinuous finite element quasi-diffusion methods*, Nuclear Science and Engineering, 191 (2018), pp. 105–120.

[2] D. Anistratov, J. Warsa, and R. Lowrie, *Spatial domain decomposition for transport problems with two-level acceleration algorithms*, (Submitted to M&C 2019).

[3] V. Gol'din, *A quasi-diffusion method of solving the kinetic equation*, USSR Comp. Math. and Math. Physics, 4 (1964), pp. 136–149.

[4] M. Miften and E. Larsen, *The quasi-diffusion method for solving transport problems in planar and spherical geometries*, Journal of Transport Theory and Statistical Physics, 22(2-3) (1992), pp. 165–186.

[5] S. Olivier, P. Maginot, and T. Haut, *High order mixed finite element discretization for the variable eddington factor equations*, (Submitted to M&C 2019).

[6] S. Olivier and J. Morel, *Variable eddington factor method for the sn equations with lumped discontinuous galerkin spatial discretization coupled to a drift-diffusion acceleration equation with mixed finite-element discretization*, Journal of Computational and Theoretical Transport, 46 (2017), pp. 480–496.

[7] J. Warsa and D. Anistratov, *Two-level transport methods with independent discretization*, Journal of Computational and Theoretical Transport, 47 (2018), pp. 424–450.

Questions?

This research is supported by the Department of Energy Computational Science Graduate Fellowship, provided under grant number DE-SC0019323 and was performed under the auspices of the U.S. Department of Energy by Los Alamos National Laboratory.

One Point Quadrature Equivalance

$$\boxed{B_1(\xi) = 1 - \xi\,,\; B_2(\xi) = \xi}$$

- Volumetric term in 1D

$$\int \frac{\mathrm{d}v}{\mathrm{d}x} E\phi \,\mathrm{d}x \to \int \frac{\mathrm{d}B_i}{\mathrm{d}x} E B_j \,\mathrm{d}x$$

- After the fact closure

$$\frac{\int \frac{\mathrm{d}B_i}{\mathrm{d}x} \int \mu^2 \psi \,\mathrm{d}\mu \,\mathrm{d}x}{\int \int \psi \,\mathrm{d}\mu \,\mathrm{d}x} \int B_j \,\mathrm{d}x = \frac{\mathrm{d}B_i}{\mathrm{d}x} \frac{\int \int \mu^2 \psi \,\mathrm{d}\mu \,\mathrm{d}x}{\int \int \psi \,\mathrm{d}\mu \,\mathrm{d}x} \int B_j \,\mathrm{d}x$$
$$= \frac{1}{2} \frac{\mathrm{d}B_i}{\mathrm{d}x} \bar{E}$$
$$\bar{E} = \frac{P_1 + P_2}{\phi_1 + \phi_2}$$

- One-point GQ: $\xi = \frac{1}{2}$, $w = 1$

$$\int \frac{\mathrm{d}B_i}{\mathrm{d}x} E B_j \,\mathrm{d}x = \frac{\mathrm{d}B_i}{\mathrm{d}x} \int E B_j \,\mathrm{d}x$$
$$= w \frac{\mathrm{d}B_i}{\mathrm{d}x} [E B_j]_{\xi = \frac{1}{2}}$$
$$= \frac{1}{2} \frac{\mathrm{d}B_i}{\mathrm{d}x} \bar{E}$$

Boundary Terms and Low-Order Quadrature

- Discrete closure for $\oint G\, u\phi \,\mathrm{d}A$ in zeroth moment's Miften-Larsen BC term

$$\oint G\mathbf{B}^T\mathbf{B}\,\mathrm{d}V \to \frac{\oint \mathbf{B}^T\mathbf{B} \int |\hat{\Omega}\cdot\hat{n}|\,\underline{\psi}\,\mathrm{d}\Omega\,\mathrm{d}A}{\oint \mathbf{B}\int \underline{\psi}\,\mathrm{d}\Omega\,\mathrm{d}A} \oint \mathbf{B}\,\mathrm{d}A$$

- Mass-matrix like term on numerator weights $|\hat{\Omega}\cdot\hat{n}|\,\psi$ non-uniformly to the nodes
- Denominator is simple average
- Evaluating G at quadrature points will never weight the numerator towards the nodes differently than the denominator

The Real G

- Casting $\nabla\vec{v} : \mathbf{E}$ as a matrix-vector product requires flattening the tensors into vectors such that

$$\langle \nabla\vec{v} \rangle \cdot \langle \mathbf{E} \rangle = \nabla\vec{v} : \mathbf{E}$$

- Order the flattened vectors as

$$\langle \nabla\vec{v} \rangle = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{bmatrix}^T , \quad \langle \mathbf{E} \rangle = \begin{bmatrix} E_{xx} & E_{xy} & E_{yx} & E_{yy} \end{bmatrix}^T$$

- Let,

$$\langle \nabla\vec{v} \rangle = \mathbf{d}\underline{v} \Rightarrow \mathbf{d} = \begin{bmatrix} \nabla B_1 & \nabla B_2 & \nabla B_3 & & & \\ & & & \nabla B_1 & \nabla B_2 & \nabla B_3 \end{bmatrix}$$

- Volumetric term is then:

$$\int \nabla\vec{v} : \mathbf{E}\,\phi\,\mathrm{d}V \rightarrow \int \mathbf{d}^T \langle \mathbf{E} \rangle \mathbf{B}\,\mathrm{d}V$$